GLOBAL EXPONENTIAL SAMPLED-DATA OBSERVERS FOR NONLINEAR SYSTEMS WITH DELAYED MEASUREMENTS

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OUTLINE

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Motivation

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Systems with a compact GAS set
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- Systems with a compact GAS set
- Conclusions
Motivation

Very frequently we meet a process with:
Motivation

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- Nonlinear characteristics,
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- Nonlinear characteristics,
- Partial measurements,
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- Nonlinear characteristics,
- Partial measurements,
- Sampled measurements,
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- Partial measurements,
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- Measurements with errors,
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Very frequently we meet a process with:

- Nonlinear characteristics,
- Partial measurements,
- Sampled measurements,
- Measurements with errors,
- Measurements with delays,
Very frequently we meet a process with:

- Nonlinear characteristics,
- Partial measurements,
- Sampled measurements,
- Measurements with errors,
- Measurements with delays,
- Uncertain sampling schedule.
Motivation

Very frequently we meet a process with:

• Nonlinear characteristics,
• Partial measurements,
• Sampled measurements,
• Measurements with errors,
• Measurements with delays,
• Uncertain sampling schedule.

In fact, we rarely meet a system without one of the above “annoying features”!!
Motivation

QUESTION

How can we design a global exponential observer for such a system?
Motivation

Very few answers!
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See also the Lyapunov Krasovskii methodologies in:
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See also the Lyapunov Krasovskii methodologies in:


For certain classes of nonlinear systems
Motivation

A general result is needed!
A General Result

Nonlinear forward complete systems of the form:

$$\dot{x} = f(x, u), x \in \mathbb{R}^n, u \in U$$  \hspace{1cm} (1)

where $U \subseteq \mathbb{R}^m$ is a non-empty set, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a smooth vector field.
A General Result

Nonlinear forward complete systems of the form:

\[ \dot{x} = f(x,u), x \in \mathbb{R}^n, u \in U \]  

(1)

where \( U \subseteq \mathbb{R}^m \) is a non-empty set, \( f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n \) is a smooth vector field.

The output is given by

\[ y = h(x) \]  

(2)

where \( h : \mathbb{R}^n \rightarrow \mathbb{R}^k \) is a smooth mapping.
Let’s put together:

\[ y(\tau_i) \approx h(x(\tau_i - r)) \]

Sampled and delayed measurements

\[ \rightarrow \text{Robust Sampled Data Observer} \]

\[ \rightarrow z(t) \approx x(t - r) \rightarrow \text{A Robust Predictor} \]

\[ \rightarrow x(t) \]
Let’s put together:

\[ y(\tau_i) \approx h(x(\tau_i - r)) \]

\[ \text{Sampled} \quad \rightarrow \quad \text{Robust} \quad \text{Observer} \]

\[ \text{delayed and} \quad \rightarrow \quad \text{Sampled Data} \]

\[ \text{measurements} \quad \rightarrow \quad z(t) \approx x(t - r) \quad \rightarrow \quad \text{Robust} \quad \text{Predictor} \]

/ \downarrow \backslash

A conventional observer
+ an intersample predictor
Assumption (H1): System (1) admits a Robust Global Exponential Observer given by
A General Result

Assumption (H1): System (1) admits a Robust Global Exponential Observer given by

\[ \dot{z} = F(z, y + ν, u) \]
\[ \dot{x} = Ψ(z) \]

where

\[ z ∈ ℝ^l, y ∈ ℝ^k, ν ∈ ℝ^k, u ∈ U ⊆ ℝ^m, \dot{x} ∈ ℝ^n \]
Assumption (H1): System (1) admits a Robust Global Exponential Observer given by

\[ \begin{align*}
\dot{z} &= F(z, y + v, u) \\
\dot{x} &= \Psi(z) \\
z &\in \mathbb{R}^l, y \in \mathbb{R}^k, v \in \mathbb{R}^k, u \in U \subseteq \mathbb{R}^m, \hat{x} \in \mathbb{R}^n
\end{align*} \tag{3} \]

where \( F : \mathbb{R}^l \times \mathbb{R}^k \times U \to \mathbb{R}^l \) and \( \Psi : \mathbb{R}^l \to \mathbb{R}^n \) are smooth vector fields, i.e., there exist a non-decreasing function \( M : \mathbb{R}_+ \to \mathbb{R}_+ \) and constants \( \sigma, \gamma > 0 \) such that for every \((x_0, z_0, u, v) \in \mathbb{R}^n \times \mathbb{R}^l \times L^\infty(\mathbb{R}_+; U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)\), the solution \((x(t), z(t))\) of (1), (2) and (3) with initial condition \((x(0), z(0)) = (x_0, z_0)\) corresponding to inputs \((u, v) \in L^\infty(\mathbb{R}_+; U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)\) exists for all \( t \geq 0 \) and satisfies the following estimate for all \( t \geq 0 \):
A General Result

Assumption (H1): System (1) admits a Robust Global Exponential Observer given by

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\begin{align*}
\dot{z} &= F(z, y + v, u) \\
\hat{x} &= \Psi(z) \\
z \in \mathbb{R}^l, y \in \mathbb{R}^k, v \in \mathbb{R}^k, u \in U \subseteq \mathbb{R}^m, \hat{x} \in \mathbb{R}^n
\end{align*}
\] (3)

where \( F: \mathbb{R}^l \times \mathbb{R}^k \times U \rightarrow \mathbb{R}^l \) and \( \Psi: \mathbb{R}^l \rightarrow \mathbb{R}^n \) are smooth vector fields, i.e., there exist a non-decreasing function \( M: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and constants \( \sigma, \gamma > 0 \) such that for every \((x_0, z_0, u, v) \in \mathbb{R}^n \times \mathbb{R}^l \times L^\infty(\mathbb{R}_+; U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)\) the solution \((x(t), z(t))\) of (1), (2) and (3) with initial condition \((x(0), z(0)) = (x_0, z_0)\) corresponding to inputs \((u, v) \in L^\infty(\mathbb{R}_+; U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)\) exists for all \( t \geq 0 \) and satisfies the following estimate for all \( t \geq 0 \):

\[
|\hat{x}(t) - x(t)| \leq e^{-\sigma t} M(|x_0| + |z_0| + \|u\|) + \gamma \sup_{0 \leq s \leq t} \left( e^{-\sigma(t-s)} |v(s)| \right)
\] (4)
A General Result

Moreover, the system
Moreover, the system

\[ \dot{z}(t) = F(z(t), w(t), u(t - r)) \]

\[ \dot{w}(t) = L_f h(\Psi(z(t)), u(t - r)) \]  (5)
Moreover, the system

\[
\begin{align*}
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\end{align*}
\]  

is forward complete for inputs \( u \in L^\infty([-r, +\infty); U) \). Furthermore, the system

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is forward complete for inputs \( u \in L^\infty([-r, +\infty); U) \). Furthermore, the system

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\]

(6)
Moreover, the system

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\dot{z}(t) &= F(z(t), w(t), u(t - r)) \\
\dot{w}(t) &= \mathcal{L} h(\Psi(z(t)), u(t - r))
\end{align*}

is forward complete for inputs $u \in L^\infty([-r, +\infty); U)$. Furthermore, the system

\begin{align*}
\dot{z}(t) &= F(z(t), w(t), u(t - r))
\end{align*}

is forward complete for inputs $(u, w) \in L^\infty([-r, +\infty); U) \times L^\infty_{\text{loc}}([-r, +\infty); \mathbb{R}^k)$. 
Assumption (H2): There exists a non-empty subset of the functions \( z : \mathbb{R}_+ \rightarrow \mathbb{R}^n \) which are absolutely continuous on every bounded interval of \( \mathbb{R}_+ \) denoted by \( A(\mathbb{R}_+;\mathbb{R}^n) \) such that system (1) admits a robust global exponential \( r \)–predictor for (1) with input \( z \in A(\mathbb{R}_+;\mathbb{R}^n) \), i.e.,
A General Result

Assumption (H2): There exists a non-empty subset of the functions $z: \mathbb{R}_+ \to \mathbb{R}^n$ which are absolutely continuous on every bounded interval of $\mathbb{R}_+$ denoted by $A(\mathbb{R}_+; \mathbb{R}^n)$ such that system (1) admits a robust global exponential $r-$predictor for (1) with input $z \in A(\mathbb{R}_+; \mathbb{R}^n)$, i.e.,

$$
\begin{align*}
\dot{\xi}(t) &= F_p(\xi_t, u_t, z(t), \dot{z}(t)) \\
\dot{\chi}(t) &= G(\xi_t, u_t, z(t)) \\
\xi(t) &\in \mathbb{R}^q, \chi(t) \in \mathbb{R}^n, u(t) \in U \subseteq \mathbb{R}^m, z(t) \in \mathbb{R}^n
\end{align*}
$$

(7)
**A General Result**

**Assumption (H2):** There exists a non-empty subset of the functions $z: \mathbb{R}_+ \to \mathbb{R}^n$ which are absolutely continuous on every bounded interval of $\mathbb{R}_+$ denoted by $A(\mathbb{R}_+; \mathbb{R}^n)$ such that system (1) admits a robust global exponential $r-$predictor for (1) with input $z \in A(\mathbb{R}_+; \mathbb{R}^n)$, i.e.,

\[
\begin{align*}
\dot{\xi}(t) &= F_p(\xi_t, u_t, z(t), \dot{z}(t)) \\
\bar{\xi}(t) &= G(\xi_t, u_t, z(t)) \\
\xi(t) &\in \mathbb{R}^q, \bar{\xi}(t) \in \mathbb{R}^n, u(t) \in U \subseteq \mathbb{R}^m, z(t) \in \mathbb{R}^n
\end{align*}
\] (7)

where $$(\xi_t)(\theta) = \xi(t + \theta), \quad (u_t)(\theta) = u(t + \theta), \quad \text{for} \quad \theta \in [-r, 0],$$

$G: C^0([-r,0]; \mathbb{R}^q) \times L^\infty([-r,0]; U) \times \mathbb{R}^n \to \mathbb{R}^n, \ F_p: C^0([-r,0]; \mathbb{R}^q) \times L^\infty([-r,0]; U) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$$
A General Result

\( \forall (x_0, \xi_0, u, z) \in C^0([-r,0]; \mathbb{R}^n) \times C^0([-r,0]; \mathbb{R}^q) \times L^\infty([-r,\infty); U) \times A(\mathbb{R}_+; \mathbb{R}^n) \) the solution \((x(t), \xi(t)) \in \mathbb{R}^n \times \mathbb{R}^k\) of (1) and (7) with initial condition \(\xi(\theta) = (\xi_0)(\theta)\), \(x(\theta) = (x_0)(\theta)\), \(\theta \in [-r,0]\), corresponding to inputs \((u, z) \in L^\infty([-r,\infty); U) \times A(\mathbb{R}_+; \mathbb{R}^n)\) is unique, defined for all \(t \geq 0\) and satisfies
A General Result

\[ \forall (x_0, \xi_0, u, z) \in C^0([-r,0]; \mathbb{R}^n) \times C^0([-r,0]; \mathbb{R}^q) \times L^\infty([-r,\infty); U) \times A(\mathbb{R}_+; \mathbb{R}^n) \quad \text{the solution} \quad (x(t), \xi(t)) \in \mathbb{R}^n \times \mathbb{R}^k \quad \text{of (1) and (7) with initial condition} \quad \xi(\theta) = (\xi_0)(\theta), \]
\[ x(\theta) = (x_0)(\theta), \quad \theta \in [-r,0], \quad \text{corresponding to inputs} \quad (u, z) \in L^\infty([-r,\infty); U) \times A(\mathbb{R}_+; \mathbb{R}^n) \]
\[ \text{is unique, defined for all} \quad t \geq 0 \quad \text{and satisfies} \]

\[ |\hat{x}(t) - x(t)| \leq e^{-\sigma t} \left( \|x_0\| + \|\xi_0\| + \|u\| + |z(0)| + \sup_{0 \leq s \leq r} \|\hat{z}(s)\| \right) \]
\[ + P \sup_{0 \leq s \leq t} \left( e^{-\sigma (t-s)} |z(s) - x(s-r)| \right), \quad \forall t \geq 0 \quad (8) \]
Moreover, for every \((z_0, u, w) \in \mathbb{R}^l \times L^\infty([-r, +\infty); U) \times L^\infty_{\text{loc}}([-r, +\infty); \mathbb{R}^k]\), the output signal \(\hat{x}(t) = \Psi(z(t))\) produced by the unique solution of (6) with initial condition \(z(0) = z_0\) and corresponding to inputs \((u, w) \in L^\infty([-r, +\infty); U) \times L^\infty_{\text{loc}}([-r, +\infty); \mathbb{R}^k]\) is a function of class \(A(\mathbb{R}_+; \mathbb{R}^n)\).
**A General Result**

**Assumption (H3):** There exist a constant $C > 0$, a continuous function $T: \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}_+$ and a non-decreasing function $N: \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $(x_0, z_0, u, v) \in \mathbb{R}^n \times \mathbb{R}^l \times L^\infty(\mathbb{R}_+; U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}_+^k)$ the solution $(x(t), z(t))$ of (1), (2) and (3) with initial condition $(x(0), z(0)) = (x_0, z_0)$ corresponding to inputs $(u, v) \in L^\infty(\mathbb{R}_+; U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}_+^k)$ satisfies the following estimate for all $t \geq T(x_0, z_0)$:
A General Result

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$$
|L_f h(\hat{x}(t), u(t)) - L_f h(x(t), u(t))| \\
\leq e^{-\sigma t} N(|x_0| + |z_0| + \|u\|) + C \sup_{0 \leq s \leq t} \left( e^{-\sigma(t-s)} \|v(s)\| \right)
$$

(9)
Theorem: Consider system (1) under hypotheses (H1-3). Let $0 < b \leq B$ be (arbitrary) constants satisfying:
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$$CB \exp(\sigma B) < 1$$

(10)
Theorem: Consider system (1) under hypotheses (H1-3). Let $0 < b \leq B$ be (arbitrary) constants satisfying:

$$CB \exp(\sigma B) < 1$$  \hspace{1cm} (10)

Then there exists a non-decreasing function $Q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for every partition $\pi = \{\tau_i\}_{i=0}^\infty$ of $\mathbb{R}_+$ with $\sup_{i \geq 0} (\tau_{i+1} - \tau_i) \leq B$ and $\inf_{i \geq 0} (\tau_{i+1} - \tau_i) \geq b$, for every $(z_0, w_0, u, v) \in \mathbb{R}^l \times \mathbb{R}^k \times L^\infty([-r, +\infty); U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)$, $(x_0, \xi_0) \in C^0([-r, 0]; \mathbb{R}^n) \times C^0([-r, 0]; \mathbb{R}^q)$ the unique solution of the system (1) with
A General Result

\[ \dot{z}(t) = F(z(t), w(t), u(t - r)) \]
\[ \hat{x}(t) = \Psi(z(t)) \]

(the conventional robust observer)
A General Result

\[
\dot{z}(t) = F(z(t), w(t), u(t - r)) \quad (\text{the conventional robust observer})
\]

\[
\hat{x}(t) = \Psi(z(t))
\]

\[
\dot{w}(t) = L_f h(\hat{x}(t), u(t - r)), \quad t \in [\tau_i, \tau_{i+1}) \quad (\text{the intersample predictor})
\]
A General Result

\[
\dot{z}(t) = F(z(t), w(t), u(t - r)) \\
\hat{x}(t) = \Psi(z(t))
\]  
(the conventional robust observer)

+ 

\[
\dot{w}(t) = L_f h(\hat{x}(t), u(t - r)), \quad t \in [\tau_i, \tau_{i+1})
\]  
(the intersample predictor)

+ 

\[
w(\tau_{i+1}) = h(x(\tau_{i+1} - r)) + v(\tau_{i+1})
\]  
(the sampled and delayed measurement)
A General Result

\[ \dot{z}(t) = F(z(t), w(t), u(t - r)) \]
\[ \hat{x}(t) = \Psi(z(t)) \quad (\text{the conventional robust observer}) \]

\[ \dot{w}(t) = L_f h(\hat{x}(t), u(t - r)), \quad t \in [\tau_i, \tau_{i+1}) \quad (\text{the intersample predictor}) \]

\[ w(\tau_{i+1}) = h(x(\tau_{i+1} - r)) + v(\tau_{i+1}) \quad (\text{the sampled and delayed measurement}) \]

\[ \dot{\xi}(t) = F_p \left( \xi_t, u_t, \hat{x}(t), \frac{d\hat{x}}{dt}(t) \right) \]
\[ \tilde{x}(t) = G(\xi_t, u_t, \hat{x}(t)) \quad (\text{the robust predictor}) \]
with initial condition \( \xi(\theta) = (\xi_0)(\theta) \), \( x(\theta) = (x_0)(\theta) \), \( \theta \in [-r, 0] \), \( (z(0), w(0)) = (z_0, w_0) \) corresponding to inputs \((u, v) \in L^\infty([-r, +\infty); U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)\) is defined for all \( t \geq 0 \) and satisfies the estimate:
A General Result

with initial condition \( \xi(\theta) = (\xi_0)(\theta), \ x(\theta) = (x_0)(\theta), \ \theta \in [-r, 0], \ (z(0), w(0)) = (z_0, w_0) \n\) corresponding to inputs \((u, v) \in L^\infty([-r, +\infty); U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)\) is defined for all \( t \geq 0 \) and satisfies the estimate:

\[
|\tilde{x}(t) - x(t)| \\
\leq e^{-\sigma t} Q \left( \|x_0\| + \|\xi_0\| + |u| + |z_0| + |w_0| + \sup_{0 \leq s \leq t} (|v(s)|) \right), \ \forall t \geq 0 \\
+ \frac{\gamma P \exp(\sigma B)}{1 - CB \exp(\sigma B)} \sup_{0 \leq s \leq t} \left( e^{-\sigma (t-s)} |v(s)| \right)
\]
Globally Lipschitz Systems

Assume:
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There exists a constant $L > 0$ such that

$$|f(x,u) - f(z,u)| \leq L|x - z|, \forall x, z \in \mathbb{R}^n, \forall u \in U$$
Globally Lipschitz Systems

Assume:

There exists a constant $L > 0$ such that

$$|f(x,u) - f(z,u)| \leq L|x - z|, \forall x, z \in \mathbb{R}^n, \forall u \in U$$

There exists a symmetric, positive definite matrix $P \in \mathbb{R}^{n \times n}$, a constant $q > 0$ and matrices $K \in \mathbb{R}^{n \times k}$, $H \in \mathbb{R}^{k \times n}$ such that:
Assume:

There exists a constant $L > 0$ such that

$$|f(x,u) - f(z,u)| \leq L|x - z|, \forall x, z \in \mathbb{R}^n, \forall u \in U$$

There exists a symmetric, positive definite matrix $P \in \mathbb{R}^{n \times n}$, a constant $q > 0$ and matrices $K \in \mathbb{R}^{n \times k}$, $H \in \mathbb{R}^{k \times n}$ such that:

$$h(x) = Hx, \forall x \in \mathbb{R}^n \text{ (linear output mapping)}$$
Globally Lipschitz Systems

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There exists a constant $L > 0$ such that

$$|f(x,u) - f(z,u)| \leq L|x - z|, \forall x, z \in \mathbb{R}^n, \forall u \in U$$

There exists a symmetric, positive definite matrix $P \in \mathbb{R}^{n \times n}$, a constant $q > 0$ and matrices $K \in \mathbb{R}^{n \times k}$, $H \in \mathbb{R}^{k \times n}$ such that:

$$(z - x)'P(f(z,u) - f(x,u)) + (z - x)'PKH(z - x) \leq -q|z - x|^2,$$

$\forall x, z \in \mathbb{R}^n, \forall u \in U$ (a conventional high-gain observer)
Globally Lipschitz Systems

**Theorem:** Let $r > 0$ be a constant. For every positive integer $p > 0$ with $Lr < p$, for every $\mu > 0$, $0 < b \leq B$ with $L|H|\sqrt{K'PPK} \sqrt{P}B < 1$, there exist a non-decreasing function $Q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and constants $\sigma, \Gamma > 0$ such that for every partition $\pi = \{\tau_i\}_{i=0}^{\infty}$ of $\mathbb{R}_+$ with $\sup_{i \geq 0} (\tau_{i+1} - \tau_i) \leq B$ and $\inf_{i \geq 0} (\tau_{i+1} - \tau_i) \geq b$, for every $(z_0, w_0, u, v) \in \mathbb{R}^n \times \mathbb{R}^k \times L^\infty([-r, +\infty); U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)$, $x_0 \in C^1([-r, 0]; \mathbb{R}^n)$, $\xi_{i,0} \in C^0([-r, 0]; \mathbb{R}^n)$ ($i = 1, \ldots, p$) the unique solution of the system (1) with
Globally Lipschitz Systems

\[ \dot{z}(t) = f(z(t), u(t-r)) + K(Hz(t) - w(t)) \]  (the conventional observer)
Globally Lipschitz Systems

\[ \dot{z}(t) = f(z(t), u(t-r)) + K(Hz(t) - w(t)) \] (the conventional observer)

\[ \hat{w}(t) = Hf(z(t), u(t-r)) \quad t \in [\tau_i, \tau_{i+1}) \] (the intersample predictor)
Globally Lipschitz Systems

\[
\dot{z}(t) = f(z(t), u(t-r)) + K(Hz(t) - w(t))
\]  
(the conventional observer)

\[
\dot{w}(t) = Hf(z(t), u(t-r)), \quad t \in [\tau_i, \tau_{i+1})
\]  
(the intersample predictor)

\[
w(\tau_{i+1}) = Hx(\tau_{i+1} - r) + v(\tau_{i+1})
\]  
(the sampled and delayed measurement)
Globally Lipschitz Systems

and the robust cascade predictor
Globally Lipschitz Systems

and the robust cascade predictor

\[
\dot{\xi}_1(t) = f(\xi_1(t), u(t - r + \delta)) - f(\xi_1(t - \delta), u(t - r)) \\
+ \dot{z}(t) - \mu \left( \xi_1(t) - z(t) - \int_{t-\delta}^{t} f(\xi_1(s), u(s - r + \delta)) ds \right)
\]

\[
\dot{\xi}_i(t) = f(\xi_i(t), u(t - r + i\delta)) - f(\xi_i(t - \delta), u(t - r + (i-1)\delta)) \\
+ \dot{\xi}_{i-1}(t) - \mu \left( \xi_i(t) - \xi_{i-1}(t) - \int_{t-\delta}^{t} f(\xi_i(s), u(s - r + i\delta)) ds \right)
\]

, \ i = 2, ..., p
with $\delta := p^{-1} r$, initial condition $x(\theta) = x_0(\theta)$, $\xi_i(\theta) = \xi_{i,0}(\theta)$, $\theta \in [-r, 0]$, $(i = 1, \ldots, p)$,

$(z(0), w(0)) = (z_0, w_0)$ corresponding to inputs $(u, v) \in L^\infty([-r, +\infty); U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)$ is defined for all $t \geq 0$ and satisfies the estimate:
Globally Lipschitz Systems

with \( \delta := p^{-1} r \), initial condition \( x(\theta) = x_0(\theta), \xi_i(\theta) = \xi_{i,0}(\theta), \theta \in [-r,0], (i = 1,\ldots,p), \) \((z(0),w(0)) = (z_0,w_0)\) corresponding to inputs \((u,v) \in L^\infty([-r,\infty); U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)\) is defined for all \( t \geq 0 \) and satisfies the estimate:

\[
\begin{align*}
|\xi_p(t) - x(t)| \\
\leq e^{-\sigma t} Q \left( \|x_0\| + \sum_{i=1}^{p} \|\xi_{i,0}\| + \|u\| + |z_0| + |w_0| + \sup_{0 \leq s \leq t} (|v(s)|) \right), \quad \forall t \geq 0 \\
+ \Gamma \sup_{0 \leq s \leq t} \left( e^{-\sigma (t-s)} |v(s)| \right)
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\]

Arbitrarily large delay!
Globally Lipschitz Systems

As \( r \to +\infty \) (the measurement delay)

then \( \Gamma \to +\infty \) (sensitivity to measurement error)

(expected)
Suppose that:

The set $U \subseteq \mathbb{R}^m$ is compact and there exist a non-empty compact set $S \subseteq \mathbb{R}^n$, a continuous function $T : \mathbb{R}^n \to \mathbb{R}_+$ and a smooth positive function $\psi : \mathbb{R}^n \to (0, +\infty)$ such that for every $u \in L^\infty(\mathbb{R}_+; U)$ and for every initial condition $x(0) \in \mathbb{R}^n$ the solution of (1) satisfies:
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$$x(t) \in S, \forall t \geq T(x(0))$$
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\[
x(t) \in S, \forall t \geq T(x(0))
\]

\[
|x(t)| \leq \psi(x(0)), \forall t \geq 0
\]
Suppose also that \( h(\mathbb{R}^n) = \mathbb{R} \) and that:

We have robust global exponential observer such that for every \((u, w) \in L^\infty([-r, +\infty); U) \times L^\infty_{loc}([-r, +\infty); \mathbb{R}^k)\) and for every initial condition \(z(0) \in \mathbb{R}^n\) it holds that:
Systems with a Compact GAS set

Suppose also that $h(\mathbb{R}^n) = \mathbb{R}$ and that:

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(a systematic procedure for designing such observers can be found in Karafyllis, I. and C. Kravaris, “Global Exponential Observers for Two Classes of Nonlinear Systems”, Systems and Control Letters, 61(7), 2012, 797-806.)
Notice that there exists a constant \( G \geq 0 \) such that for every 
\((x_0, z_0, u, v) \in \mathbb{R}^n \times \mathbb{R}^l \times L^\infty(\mathbb{R}^+; U) \times L^\infty_{loc}(\mathbb{R}^+; \mathbb{R}^k)\) the solution \((x(t), z(t))\) of (1), (2) and (3) with initial condition \((x(0), z(0)) = (x_0, z_0)\) corresponding to inputs 
\((u, v) \in L^\infty(\mathbb{R}^+; U) \times L^\infty_{loc}(\mathbb{R}^+; \mathbb{R}^k)\) satisfies the following estimate for all 
\(t \geq \max(T(x_0), T(z_0))\):
Notice that there exists a constant $G \geq 0$ such that for every $(x_0, z_0, u, v) \in \mathbb{R}^n \times \mathbb{R}^l \times L^\infty(\mathbb{R}_+; U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)$ the solution $(x(t), z(t))$ of (1), (2) and (3) with initial condition $(x(0), z(0)) = (x_0, z_0)$ corresponding to inputs $(u, v) \in L^\infty(\mathbb{R}_+; U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)$ satisfies the following estimate for all $t \geq \max(T(x_0), T(z_0))$:

$$|L_fh(\hat{x}(t), u(t)) - L_fh(x(t), u(t))| \leq G|\hat{x}(t) - x(t)|$$
Systems with a Compact GAS set

Define:
Systems with a Compact GAS set

Define:

\[ p(s) := \max \{ |f(\xi, u)| : u \in U, |\xi| \leq K \max \{ \psi(z) : |z| \leq s \} \} \]
Systems with a Compact GAS set

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\[ \tilde{S} := \{ \xi \in \mathbb{R}^n : |\xi| \leq 1 + a + r \, p(a) \} \]
Systems with a Compact GAS set

Define:

\[
p(s) := \max \left\{ f(\xi, u) : u \in U, |\xi| \leq K \max \left\{ \psi(z) : |z| \leq s \right\} \right\}
\]

\[
a := \max \{|z| : z \in S\}
\]

\[
\tilde{S} := \left\{ \xi \in \mathbb{R}^n : |\xi| \leq 1 + a + r p(a) \right\}
\]

Let \( G_1, G_2 > 0 \) be constants satisfying
Systems with a Compact GAS set

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Let \( G_1, G_2 > 0 \) be constants satisfying

\[ \left| f \left( q \left( \frac{|\xi|}{\psi(z)} \right) \xi, u \right) - f \left( q \left( \frac{|x|}{\psi(y)} \right) x, u \right) \right| \leq G_1 |\xi - x| + G_2 |z - y| \]

\[ \forall u \in U, y, x, z \in S, \xi \in \tilde{S} \]
Theorem: If \( G_1 r < 1 \), then for every \( \mu > 0 \), \( 0 < b \leq B \) with \( G \gamma B < 1 \), where \( \gamma \) is the gain of the measurement error for the conventional observer (3), there exist a non-decreasing function \( Q : \mathbb{R}_+ \to \mathbb{R}_+ \) and constants \( \sigma, \Gamma > 0 \) such that for every partition \( \pi = \{ \tau_i \}_{i=0}^\infty \) of \( \mathbb{R}_+ \) with
\[
\sup_{i \geq 0} (\tau_{i+1} - \tau_i) \leq B \quad \text{and} \quad \inf_{i \geq 0} (\tau_{i+1} - \tau_i) \geq b, \quad \text{for every}
\]
\((z_0, w_0, u, v) \in \mathbb{R}^n \times \mathbb{R}^k \times L^\infty([-r, +\infty); U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k),\)
\((x_0, \xi_0) \in C^0([-r, 0]; \mathbb{R}^n) \times C^0([-r, 0]; \mathbb{R}^n) \) the solution of (1) with
Systems with a Compact GAS set

\[ \dot{z}(t) = F(z(t), w(t), u(t - r)) \]
\[ \hat{x}(t) = \Psi(z(t)) \]  
(the conventional observer)
Systems with a Compact GAS set

\[
\begin{align*}
\dot{z}(t) &= F(z(t), w(t), u(t - r)) \\
\hat{x}(t) &= \Psi(z(t)) \\
\dot{w}(t) &= L_f h(\hat{x}(t), u(t - r)), \ t \in [\tau_i, \tau_{i+1})
\end{align*}
\]

(the conventional observer) (the inter-sample predictor)
Systems with a Compact GAS set

\[
\begin{align*}
\dot{z}(t) &= F(z(t), w(t), u(t - r)) \\
\hat{x}(t) &= \Psi(z(t))
\end{align*}
\]
(the conventional observer)

\[
\dot{w}(t) = L_f h(\hat{x}(t), u(t - r)), \quad t \in [\tau_i, \tau_{i+1})
\]
(the intesample predictor)

\[
w(\tau_{i+1}) = h(x(\tau_{i+1} - r)) + v(\tau_{i+1})
\]
(the sampled and delayed measurement)
Systems with a Compact GAS set

\[ \dot{z}(t) = F(z(t), w(t), u(t - r)) \]  
\[ \dot{x}(t) = \Psi(z(t)) \]  
\[ \dot{w}(t) = L_f h(\hat{x}(t), u(t - r)), \ t \in [\tau_i, \tau_{i+1}) \]  
\[ w(\tau_{i+1}) = h(x(\tau_{i+1} - r)) + v(\tau_{i+1}) \]

\[ \dot{\xi}(t) = \dot{z}(t) + \frac{d}{dt} \int_{t-\delta}^{t} f \left( q \left( \frac{|\xi(s)|}{\psi(z(t))} \right) \xi(s), u(s - r + \delta) \right) ds \]  
\[ - \mu \left( \xi(t) - z(t) - \int_{t-\delta}^{t} f \left( q \left( \frac{|\xi(s)|}{\psi(z(t))} \right) \xi(s), u(s - r + \delta) \right) ds \right) \]  
\[ \text{(the predictor)} \]
Systems with a Compact GAS set

with \( \delta = r \), initial condition \( x(\theta) = x_0(\theta), \quad \xi(\theta) = \xi_0(\theta), \quad \theta \in [-r,0], \)

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Systems with a Compact GAS set

with $\delta = r$, initial condition $x(\theta) = x_0(\theta)$, $\xi(\theta) = \xi_0(\theta)$, $\theta \in [-r, 0]$, $(z(0), w(0)) = (z_0, w_0)$ corresponding to inputs $(u, v) \in L^\infty([-r, +\infty); U) \times L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^k)$ is defined for all $t \geq 0$ and satisfies the estimate:

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|\xi(t) - x(t)| \\
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Systems with a Compact GAS set

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$$
|\xi(t) - x(t)| \\
\leq e^{-\sigma t} Q\left(\left\|x_0\right\| + \left\|\xi_0\right\| + \|u\| + |z_0| + |w_0| + \sup_{0 \leq s \leq t} (|v(s)|)\right), \forall t \geq 0
$$

$$
+ \Gamma \sup_{0 \leq s \leq t} \left(e^{-\sigma(t-s)} |v(s)|\right)
$$

Not arbitrarily large delay
Novel results for a practical and mathematically challenging problem!
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Possible use for predictor-based control of systems with delayed input
Conclusions

Novel results for a practical and mathematically challenging problem!

Possible use for predictor-based control of systems with delayed input

Possible extensions to other classes of nonlinear systems
THANK YOU!