Non-Asymptotic estimation for online systems
Finite-time algorithms and applications

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OUTLINE

1. Non-A and finite time
2. Algebra
3. Homogeneity
4. Applications
5. Take-home message
Team

Non-A

- Inria project-team (jointly with CNRS, EC Lille, Univ.Lille1)
- created July 2012 (initiated as a team in 2011)
- located in Lille (+ Nancy + Paris + Reims)
- 25 people from 11 countries (13 permanent, 6 PhD, 3 Post-Doc, 3 Eng.)
General positioning

Non-Asymptotic estimation for online systems
→ *a closer reading?*

→ *real-time signal, closed-loop control*
General positioning

Non-Asymptotic estimation for online systems

→ finite-time algorithms

(for specification, certification, separation...)

Non-Asymptotic estimation for online systems

→ infinitely many works!

- parameters (identification)
- states (observation)
- derivatives (differentiation)
- inputs (left inversion)
- noisy data (filtering)
General positioning

General classification: (estimation techniques)

- output-based (model-free) estimation/differentiation
- numerical differentiation
  - finite diff. [Khan-Qu-Ramm...], Fourier transf. [Dou, Qian...], mollification [Hao, Murio...],
- digital filtering
  - [France: Chaplais, Diop... Abroad: AlAliou, Chen, Grizzle, Jackson, Lee, Mullis, Rader, Roberts...]
- model-based estimation
  - [France: Besançon, Chitour, Gauthier, Glumineau-Moog-Flestat, Hameiri, Ibrir, Martin-Rouchon, Praly...]
  - Abroad: Alligöwer, Astolfi, Drakunov, Kazantzis-Kravaris, Khalil, Krener, Kupka, Li, Qian, Zeitz...]

Finite-time convergence:

- Hybrid/discontinuous ⇒ Nonsmooth/Sliding observers
  - [Barolini et al., Brogliato et al., Drakunov, Levant, Edwards, Spurgeon...]
- Continuous systems ⇒ Homogeneous observers
  - [Andreifl-Praly-Astolfi, Moulay-Perruquetti, Shen-Xia...]

Focus on two new standpoints for finite-time algorithms:

- Algebra: Fliess, Sira-Ramirez - ESAIM COCV 2003
  An algebraic framework for linear identification
- Homogeneity: Perruquetti, Floquet, Moulay - IEEE TAC 2008
  Finite-time observers: application to secure communication

 usable for
- numerical differentiation
- model-based observation
- identification or detection
- filtering...
2 ALGEBRAIC TECHNIQUES
An introduction with examples

2.1 Algebraic techniques: simple examples
→ double differentiation

Estimate $y(t) \rightarrow \frac{d^2y}{dt^2}$ over a sliding window with small size $T$:

$$y(t) = y(0) + y^{(1)}(0)t + \frac{1}{2} y^{(2)}(0)t^2, \ t \in [0, T]$$

$$\Rightarrow y(s) = \frac{1}{s}y(0) + \frac{1}{s^2} y^{(1)}(0) + \frac{1}{s^3} y^{(2)}(0)$$

Apply the annihilating operator: $\frac{1}{s^3} \frac{d^2}{ds^2} s^2$:

$\times s^2$: $s^2 \ddot{y} = s y(0) + y^{(1)}(0) + \frac{1}{s} y^{(2)}(0)$

$\frac{d^2}{ds^2}$: $\Rightarrow 2\ddot{y} + 4s \dot{y} + s^2 \frac{\ddot{y}}{s} = \frac{2}{s^3} y^{(2)}(0)$

$\times \frac{1}{s^3}$: $\Rightarrow y^{(2)}(0) = \frac{5!}{T^5} \int_0^T \left( \tau^2 - 4(T - \tau) + (T - \tau)^2 \right) y(\tau) d\tau$
Algebraic techniques: simple examples

→ differentiation: a look on algebra

This derivative estimation involves the differential operator:

\[ \Pi = \frac{1}{s^3} \frac{d^2}{ds^2} s^2 \in \mathbb{R}(s) \left[ \frac{d}{ds} \right] \]

→ defining a Weyl Algebra structure ⇒ canonical form:

\[ \Pi = \frac{1}{s} \frac{d^2}{ds^2} + \frac{1}{s^2} \frac{d}{ds} + \frac{2}{s^3} \]

right \[ \mathbb{R}(s) \left[ \frac{d}{ds} \right] \]

left \[ \mathbb{R}(s) \]

Recall that the Weyl Algebra is non commutative: let \( p = \frac{d}{ds} \) and \( q = s \times \)

then the commutator \([p, q] = pq - qp = 1\)

R. Ushirobira, W. Perruquetti, M. Mboup, M. Fliess. IFAC SysId ’12, Brussels, 2012

Algebraic techniques: simple examples

→ identification (simplest case!)

\[ \hat{y}(t) = \hat{\gamma} y(t) + u(t) + \gamma_0 \]

\[ s\hat{\gamma}(s) - \gamma(0) = a\hat{\gamma}(s) + \hat{u}(s) + \frac{\gamma_0}{s} \]

× s:

\[ s^2\hat{\gamma}(s) - sy(0) = a s\hat{\gamma}(s) + s\hat{u}(s) + \gamma_0 \]

\[ \frac{d^2}{ds^2} \left( -s \frac{d^2}{ds^2} \hat{y}(s) - 2 \frac{d}{ds} \hat{y}(s) \right) a = -2 s \hat{\gamma}(s) - 4 s \frac{d}{ds} \hat{y}(s) - s^2 \frac{d^2}{ds^2} \hat{y}(s) + 2 \frac{d^2}{ds^2} \hat{u}(s) + s \frac{d^2}{ds^2} \hat{\gamma}(s) \]

× \[ \frac{1}{s^3} \]

\[ \gamma = -\frac{1}{2} \int_0^1 (6\tau^2 - 6\tau + 1) y(t\tau)d\tau + \int_0^1 \tau(\tau - 1)(2\tau - 1)u(t\tau)d\tau \]

(toolbox)
Algebraic techniques: simple examples

→ frequency estimation for noisy periodic signals

\[ y(t) = a_1 \exp(i\omega_1 t + \phi_1) + a_2 \exp(i\omega_2 t + \phi_2) + \beta + \omega \]

Algebraic techniques: less simple examples

→ frequency estimation for noisy periodic signals

sum of two sine functions + mean value + noise

\[ z(t) = \omega(t) \text{ for } \omega = 0 \]
\[ z(t) - i(\omega_1 + \omega_2) z(t) - i\omega_1 z(t) - \beta = 0 \]
\[ (s^2 + \theta_2 s^2 + \theta_1 s) Z(s) + (s^2 + \theta_2 s + \theta_1) \theta_2 = 0 \]
Algebraic techniques: less simple examples

→ extension to differentiation of multi-variate signals

Irregular grids, n-D noisy signal
(here, SNR 25dB)

Figure 3: A slice of the noisy surface at \( z = 0 \) and \( -2 < y < 2 \), SNR 25dB

Simultaneous identification of \( a \) and \( \tau \)

\[
\dot{y}(t) + ay(t) = y(0)\delta + \gamma_0 H + bu(t - \tau)
\]

\( y(0) = 0.3, a = 2, \tau = 0.6, \gamma_0 = 2, b = 1, u_0 = 1. \)

Algebraic techniques: less simple examples

→ delay system (real, approx. 2nd order)

\[ G(s) = \frac{0.84 e^{-0.13s}}{0.09 s^2 + 0.55 s + 1} \]
\[ = \frac{K e^{-\tau s}}{\alpha_2 s^2 + \alpha_1 s + 1} \]
\[ \tau \approx 0.8 \text{ s} \]

Algebraic techniques: less simple examples

→ switched system

\[ \dot{y} = A y = k u, \quad a(t) = \sum_{i} a_i \chi_{[\tau_i, \tau_{i+1}]}(t) \]
\[ \dot{\theta}(t_0, \theta) = k \int_{t_0}^{t} \theta_0 d\theta - t_0 + t_0 \theta(t_0) + \int_{t_0}^{t} y d\theta \]
\[ \chi(X) = 0 \text{ or } 1, \text{ characteristic function of the set } X \]
Algebraic techniques: less simple examples

→ impulsive systems

\[ \sum_{j=0}^{\infty} a_j g_j(u, y) = \psi_0 + \sum_{j=1}^{\infty} b_j \delta(t-t_j). \]

The thermostat as a hybrid system

Bouncing ball / Rocking block

\[ \frac{d^2 y}{dt^2} = f(y) = \begin{cases} \frac{g}{\alpha} \sin(\alpha (1 - y \text{sgn}(y))) & \text{(bouncing ball)} \\ \frac{dy}{dt}(t_k) = \frac{dy}{dt}(t_{k-1}), \quad k = 1, 2, \ldots & \text{(rocking block)} \end{cases} \]

Algebraic techniques:

→ some references – pdf on http://hal.archives-ouvertes.fr/

- **algebra**

- **identification**

- **differentiation**

- **diagnosis**

- **delays & switches**

- **impulses**
Homogeneity and finite time

Again, very simple...

\[ \dot{x} = g(x) = -\sqrt{|x|} \text{sign}(x) \]

For any \( \lambda > 0 \) and some \( d \) (here \( d = \frac{1}{2} \)):

\[ g(\lambda x) = \lambda^d g(x) \]

\( g \) is said to be \textit{homogeneous} of degree \( d \).

Note that:

\begin{itemize}
\item \( g \in C^0 \) but is not Lipschitz (\( 0 < d < 1 \))
\item solutions \( x(t) \) have a \textit{finite-time convergence}
\end{itemize}

\[ x_0 \geq 0 \rightarrow x(t) = (\sqrt{x_0} - \frac{t}{2})^2 \Rightarrow T = 2\sqrt{x_0} \]
Homogeneity and finite time

A bit more general...

Definition
For a function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), if for any (positive) constant \( \lambda \) and all \( x \in \mathbb{R}^n \)

\[
    f(\lambda x) = \lambda^m f(x),
\]

then the function \( f \) is called (positively) homogeneous with degree \( m \).

Theorem (Euler’s theorem on homogeneous functions)
Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a \( C^1 \) homogeneous function of degree \( m \), then

\[
    \frac{df(x)}{dx} = mf(x).
\]

Homogeneity and finite time

A homogeneous finite-time differentiator

\[
\begin{align*}
    \dot{\xi}_1 &= -k_1|\xi_1 - y|^{\alpha_1}\text{sign}(\xi_1 - y) + \xi_2, \\
    \dot{\xi}_2 &= -k_2|\xi_1 - y|^{\alpha_2}\text{sign}(\xi_1 - y) + \xi_3, \\
    &\vdots \\
    \dot{\xi}_n &= -k_n|\xi_1 - y|^{\alpha_n}\text{sign}(\xi_1 - y).
\end{align*}
\]

with \( 0 < \alpha_i < 1 \)


Advantages of homogeneous systems:
- local = global
- GAS = ISS (measurement noise robustness, additive disturbances compensation)
- finite-time stability / fixed-time stability
Homogeneity and finite time

Ongoing developments

1. How to generalize homogeneity to larger classes of systems?
   (ex: with delay)
   Denis Efimov and Wilfrid Perruqueti. Homogeneity for time-delay systems.
   In IFAC WC 2011, pages 1–6, Milano, Italy, August 2011.

2. "Finite-time" generally depends on the initial condition.
   Can one achieve $T$ independently of $x_0$?
   $\rightarrow$ Fixed-time convergence
   Andrey Polyakov. Nonlinear feedback design for fixed-time stabilization
   of Linear Control Systems. IEEE Transactions on Automatic Control,
   57(8):2106–2110, August 2012.

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Homogeneity and finite time

Few words on fixed time stabilization

\[
\begin{align*}
\text{Asymptotic stability (Lyapunov 1892):} & \quad u(t) = -x(t) \\
& \quad x(t) = e^{-t}x_0 \rightarrow 0 \text{ if } t \rightarrow +\infty
\end{align*}
\]

\[
\begin{align*}
\text{Finite-time stability (Roxin 1966):} & \quad u(t) = -\text{sign}[x(t)] \\
& \quad x(t) = 0 \text{ for } t \geq \|x_0\|
\end{align*}
\]

\[
\begin{align*}
\text{Fixed-time stability (Polyakov 2012):} & \quad u(t) = -\left(|x(t)|^{1/2} + |x(t)|^{3/2}\right)\text{sign}[x(t)] \\
& \quad x(t) = 0 \text{ for } t \geq \pi \\
& \quad \text{independently of } x_0
\end{align*}
\]
Homogeneity and finite time

Additional motivations for fixed-time stabilization

I. Robustness (Pervozvanski 1971)

\[ \dot{x} = \lambda x + u \]

where \( x \in \mathbb{R} \) - state, the number \( \lambda \in \mathbb{R} \) is unknown, \( u \in \mathbb{R} \) - control. For

\[ u = -\mu x^3 \quad \mu > 0 \]

we have

if \( \lambda > 0 \) then \( x \to \pm \sqrt{\lambda/\mu} \) as \( t \to +\infty \)  (practical stab.)

if \( \lambda \leq 0 \) then \( x \to 0 \) as \( t \to +\infty \)  (asympt. stab.)

II. Real-life applications for automobile engine control in GMC (Kolyubin, Efimov et al. 2011)

\[ u = -\alpha x - \beta \text{sign}[x] - \gamma x^2 \quad \text{« nonlinear PID-controller»} \]

Homogeneity and finite time

Some references

- **homogeneity and finite-time observers**
  T. Menard, E. Moulay, W. Perruquetti. A global high-gain finite time observer. IEEE TAC, 55 (6): 1500-1506, 2010

- **homogeneity and ISS**
  E. Benua, A. Polyakov, D. Efimov, W. Perruquetti. On ISS and iISS properties of homogeneous systems. ECC’13, Zurich, July 2013

- **homogeneity and finite-time control**
  E. Benua, W. Perruquetti, D. Efimov, E. Moulay. Finite-time output stabilization of the double integrator. CDC’12, Maui, December 2012

- **homogeneity and delay**
  D. Efimov, W. Perruquetti. Homogeneity for time-delay systems. IFAC’11, Milano, August 2011

- **fixed-time stabilization**
Applications
2nd order differentiation w.r.t. time (angle of a pendulum)
Applications

Ball and Beam system: 1\textsuperscript{st} order differentiation w.r.t. time

\begin{itemize}
  \item ‘real’ angle speed $\theta^{(1)}$
  \item algebraic Jacobi estimator
  \item high-gain observer
  \item sliding-mode differentiator
\end{itemize}

\begin{equation}
  SNR = 24.5\text{dB} \text{ and } T_x = 10^{-4}
\end{equation}

Applications

Human posture in the sagittal plane using... accelerometer

Goal: minimize the number of inertial sensor required to describe squat task, with the purpose of developing simple, low cost and daily life evaluation and rehabilitation tools.

Existing result: [Bonnet et al., 2012] proposes to estimate ankle, knee and hip joints during squat exercise using on single inertial sensor with an offline least-square estimation process.
Applications

Human posture in the sagittal plane using... accelerometer

on-line algebraic estimation : Perruquetti et al. CDC 2012

Applications
General context: Wireless Sensor and Actuator Networks (WSAN)
deploy wireless sensor networks
by means of collaborating mobile robots

FIT:
Future Internet of Things
EquipEx national program
with Inria FUN

Hundreds of robots collaborating with thousands of sensors
Applications
WSAN: focus on localization

\[ \tan \theta = \frac{\dot{y}}{\dot{x}} \]

→ 1 landmark + 1 target is enough

H. Sert, A. Kokosy, and W. Perruquetti.
A single landmark based localization algorithm for non-holonomic mobile robots.

Localizability of unicycle mobile robots: an algebraic point of view.

Applications
Some references

- localization

- ball and beam system

- position from accelerometer
Fast differentiation algorithms
→ delayed derivatives
→ with small delay
→ with known delay (algebra)

+ robust to noise
+ deterministic
+ feasible online

Various fields are concerned:
→ signal processing
→ identification, observation
→ mode detection
→ control

Take-home message

→ toolbox

\[ y(t-T) \]
\[ \frac{dy}{dt} \]
\[ \frac{d^2y}{dt^2} \]
\[ \ldots \]
\[ \frac{d^ny}{dt^n} \]